Solution 1

1. A finite trigonometric series is of the form $a_0+\sum_{n=1}^N(a_n\cos nx+b_n\sin nx)$. A trigonometric polynomial is of the form $p(\cos x,\sin x)$ where $p(x,y)$ is a polynomial of two variables x,y. Show that a function is a trigonometric polynomial if and only if it is a finite Fourier series.

   Solution Let
$$p(x,y)=\sum_{j,k,\ 1\le j+k\le N}^{N} a_{jk}x^jy^k$$
be a polynomial of degree N. A general trigonometric polynomial is of the form
$$p(\cos x,\sin x)=\sum_{j,k} a_{jk}\cos^j x\sin^k x\ .$$
Plugging Euler's formulas $\cos x=\frac{1}{2}(e^{ix}+e^{-ix}), \sin x=\frac{1}{2i}(e^{ix}-e^{-ix})$, into this expression, one has
$$p(\cos x,\sin x)=\sum_{j,k} a_{jk}\left(\frac{e^{ix}+e^{-ix}}{2}\right)^j\left(\frac{e^{ix}-e^{-ix}}{2i}\right)^k\ .$$
Collecting the terms into series in e^{inx},
$$p(\cos x,\sin x)=\sum_{n=-N}^{N} c_ne^{inx}\ ,$$
which is a finite Fourier series.

   Conversely, observe that $\cos 2x=\cos^2 x-\sin^2 x, \sin 2x=2\cos x\sin x$, by induction you can show that $\cos nx$ and $\sin nx$ can be expressed as $p(\cos x,\sin x)$ of degree N. Hence a finite Fourier series $f(x)=a_0+\sum_{n=1}^N(a_n\cos nx+b_n\sin nx)$ can be written as a trigonometric polynomial.

2. Let f be a 2π-periodic function which is integrable over $[-\pi,\pi]$. Show that it is integrable over any finite interval and
$$\int_I f(x)dx=\int_J f(x)dx,$$
where I and J are intervals of length 2π.

   Solution It is clear that f is also integrable on $[n\pi,(n+2)\pi],\ n\in\mathbb{Z}$, so it is integrable on any finite interval. To show the integral identity it suffices to take $J=[-\pi,\pi]$ and $I=[a,a+2\pi]$ for some real number a. Since the length of I is 2π, there exists some n such that $n\pi\in I$ but $(n+2)\pi$ does not belong to the interior of I. We have
$$\int_a^{a+2\pi} f(x)dx=\int_a^{n\pi} f(x)dx+\int_{n\pi}^{a+2\pi} f(x)dx.$$
Using
$$\int_a^{n\pi} f(x)dx=\int_{a+2\pi}^{(n+2)\pi} f(x)dx$$
(by a change of variables), we get
$$\int_a^{a+2\pi} f(x)dx=\int_{a+2\pi}^{(n+2)\pi} f(x)dx+\int_{n\pi}^{a+2\pi} f(x)dx=\int_{n\pi}^{(n+2)\pi}\ .$$

Now, using a change of variables again we get

$$\int_{n\pi}^{(n+2)\pi} f(x)dx = \int_{-\pi}^{\pi} f(x)dx.$$

3. Verify that the Fourier series of every even function is a cosine series and the Fourier series of every odd function is a sine series.

Solution Write

$$f(x) \sim a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx).$$

Suppose $f(x)$ is an even function. Then, for $n \geq 1$, we have

$$\pi b_n = \int_{-\pi}^{\pi} \sin nx f(x)dx = \int_{-\pi}^{0} \sin nx f(x)dx + \int_{0}^{\pi} \sin nx f(x)dx \ .$$

By a change of variable and using $f(-x) = f(x)$ since $f(x)$ is an even function,

$$\int_{-\pi}^{0} \sin nx f(x)dx = \int_{0}^{\pi} \sin(-nx) f(-x)dx = -\int_{0}^{\pi} \sin nx f(x)dx,$$

one has

$$\pi b_n = -\int_{0}^{\pi} \sin nx f(x)dx + \int_{0}^{\pi} \sin nx f(x)\, dx = 0.$$

Hence the Fourier series of every even function f is a cosine series.

Now suppose $f(x)$ is an odd function. Then, for $n \geq 1$, we have

$$\pi a_n = \int_{-\pi}^{\pi} \cos nx f(x)dx = \int_{-\pi}^{0} \cos nx f(x)dx + \int_{0}^{\pi} \cos nx f(x)dx \ .$$

By a change of variable and using $f(-x) = -f(x)$ since $f(x)$ is an odd function,

$$\int_{-\pi}^{0} \cos nx f(x)dx = \int_{0}^{\pi} \cos(-nx) f(-x)dx = -\int_{0}^{\pi} \cos nx f(x)dx,$$

one has

$$\pi a_n = -\int_{0}^{\pi} \cos nx f(x)dx + \int_{0}^{\pi} \cos nx f(x)dx = 0\ , \quad \forall n \geq 0\ .$$

4. Here all functions are defined on $[-\pi, \pi]$. Verify their Fourier expansion and determine their convergence and uniform convergence (if possible).

   (a)

$$x^2 \sim \frac{\pi^2}{3} - 4\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos nx,$$

   (b)

$$|x| \sim \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos(2n-1)x,$$

(c)

$$f(x) = \begin{cases} 1, & x \in [0, \pi] \\ -1, & x \in [-\pi, 0] \end{cases} \sim \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin(2n-1)x,$$

(d)

$$g(x) = \begin{cases} x(\pi - x), & x \in [0, \pi) \\ x(\pi + x), & x \in (-\pi, 0) \end{cases} \sim \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \sin(2n-1)x.$$

Solution

(a) Consider the function $f_1(x) = x^2$. As $f_1(x)$ is even, its Fourier series is a cosine series and hence $b_n = 0$.

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x^2 dx = \frac{1}{2\pi} \frac{x^3}{3} \bigg|_{-\pi}^{\pi} = \frac{\pi^2}{3},$$

and by integration by parts,

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos nx dx \\ &= \frac{1}{n\pi} x^2 \sin nx \bigg|_{-\pi}^{\pi} - \frac{2}{n\pi} \int_{-\pi}^{\pi} x \sin nx dx \\ &= \frac{2}{n^2\pi} x \cos nx \bigg|_{-\pi}^{\pi} - \frac{2}{n^2\pi} \int_{-\pi}^{\pi} \cos nx dx \\ &= 4 \frac{(-1)^n}{n^2}. \end{aligned}$$

For $n \geq 1$,

$$|a_n| = | - 4 \frac{(-1)^{n+1}}{n^2} | \leq \frac{4}{n^2}.$$

We conclude that the Fourier series converges uniformly by the Weierstrass M-test.

(b) Consider the function $f_2(x) = |x|$. As $f_2(x)$ is even, its Fourier series is a cosine series and hence $b_n = 0$.

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |x| dx = \frac{1}{2\pi} \frac{x^2}{2} \bigg|_{-\pi}^{\pi} = \frac{\pi}{2},$$

and by integration by parts,

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \cos nx dx = \frac{2}{\pi} \int_{0}^{\pi} x \cos nx dx \\ &= \frac{2}{n\pi} x \sin nx \bigg|_{0}^{\pi} - \frac{2}{n\pi} \int_{0}^{\pi} \sin nx dx \\ &= - \frac{2}{n^2\pi} \cos nx \bigg|_{0}^{\pi} \\ &= -2 \frac{[(-1)^n - 1]}{n^2\pi}. \end{aligned}$$

For $n \geq 1$,

$$|a_n| = |2 \frac{[(-1)^n - 1]}{n^2\pi}| \leq \frac{4}{\pi n^2}.$$

We conclude that the Fourier series converges uniformly by the Weierstrass M-test.

(c) As $f(x)$ is odd, its Fourier series is a sine series and hence $a_n = 0$.

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = \frac{2}{\pi}\int_0^{\pi} \sin nx dx \\ &= \frac{2}{n\pi}\cos nx\Big|_0^{\pi} \\ &= 2\frac{[(-1)^n - 1]}{n\pi}. \end{aligned}$$

Now we consider the convergence of the series $\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin(2n-1)x$. Fix $x \in (-\pi, 0)\cup(0,\pi)$, Using the elementary formula

$$\sum_{n=1}^{N}\sin(2n-1)x = \frac{\sin^2(N+1)x}{\sin x},$$

one has that the partial sums $|\sum_{n=1}^{N}\sin(2n-1)x| = |\frac{\sin^2(N+1)x}{\sin x}| \le |\frac{1}{\sin x}|$ are uniformly bounded. This also holds for $x = 0$, in which case $|\sum_{n=1}^{N}\sin(2n-1)0| = 0$. Furthermore, the coefficients $1/(2n-1)$ decreases to 0. We conclude that the Fourier series converges pointwisely by Dirichlet's test.

(d) As $g(x)$ is odd, its Fourier series is a sine series and hence $a_n = 0$. By integration by parts,

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\sin nx dx = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\sin nx dx \\ &= -\frac{2}{n\pi}x(\pi - x)\cos nx\Big|_0^{\pi} + \frac{2}{n\pi}\int_0^{\pi}(\pi - 2x)\cos nx dx \\ &= \frac{2}{n^2\pi}(\pi - 2x)\sin nx\Big|_0^{\pi} + \frac{4}{n^2\pi}\int_0^{\pi}\sin nx dx \\ &= -\frac{4}{n^3\pi}\cos nx\Big|_0^{\pi} \\ &= -\frac{4}{n^3\pi}[(-1)^n - 1]. \end{aligned}$$

As

$$|b_n| \le \frac{8}{\pi n^3},$$

we conclude that the Fourier series converges uniformly by the Weierstrass M-test.

5. Show that

$$x^2 \sim \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{\cos nx}{n^2} - 4\pi\sum_{n=1}^{\infty}\frac{\sin nx}{n},$$

for $x \in [0, 2\pi]$. Compare it with 4(a).

Solution It shows that a function may have two different Fourier expansions over a subinterval. Here we have two such expansions over $[0, \pi]$.

Consider the function $f(x) = x^2$.

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} x^2 dx = \frac{1}{2\pi}\frac{x^3}{3}\Big|_0^2 = \frac{4\pi^2}{3},$$

and by integration by parts,

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{2\pi} x^2 \cos nx dx \\ &= \frac{1}{n\pi}x^2 \sin nx\Big|_0^{2\pi} - \frac{1}{n\pi}\int_0^{2\pi} x \sin nx dx \\ &= \frac{2}{n^2\pi}x\cos nx\Big|_0^{2\pi} - \frac{2}{n^2\pi}\int_0^{2\pi} \cos nx dx \\ &= \frac{4}{n^2}, \end{aligned}$$

and

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_0^{2\pi} x^2 \sin nx dx \\ &= -\frac{1}{n\pi}x^2 \cos nx\Big|_0^{2\pi} + \frac{2}{n\pi}\int_0^{2\pi} x \cos nx dx \\ &= -\frac{4\pi}{n} + \frac{2}{n^2\pi}x\sin nx\Big|_0^{2\pi} - \frac{2}{n^2\pi}\int_0^{2\pi} \sin nx dx \\ &= -\frac{4\pi}{n}. \end{aligned}$$

Remark. For a function f defined on $[0, 2\pi]$, its Fourier series is given by

$$f \sim a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx),$$

where

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx\ ,\ a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx\ ,$$

and

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx\ .$$

The reason is similar to what we did for functions on $[-T, T]$. The function $g(x) = f(x+\pi)$ is defined on $[-\pi, \pi]$. Then

$$f(x+\pi) = g(x) \sim \alpha_0 + \sum_{n=1}^{\infty}(\alpha_n \cos nx + \beta_n \sin nx)\ .$$

By writing everything in terms of f, we get the formulas. Can you write down the formula of the Fourier coefficients for a function on $[a, b]$?

6. Find the Fourier series of the function $|\sin x|$ on $[-\pi, \pi]$.

Solution. The function $|\sin x|$ is even. Using formulas such as

$$a_n = \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx\,dx,$$

we get

$$a_n = -\frac{2}{\pi}\,\frac{(-1)^n + 1}{n^2 - 1},\ n \ge 1,\ a_0 = \frac{2}{\pi}\ ,$$

and

$$|\sin x| \sim \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{\cos 2x}{3} + \frac{\cos 4x}{15} + \frac{\cos 6x}{35} + \cdots\right) .$$

7. Let f be a 2π-periodic function whose derivative exists and is integrable on $[-\pi, \pi]$. Show that its Fourier coefficients decay to 0 as $n \to \infty$ without appealing to Riemann-Lebesgue lemma. Hint: Use integration by parts to relate the Fourier coefficients of f to those of f'.

   Solution Let a_n', b_n' be the Fourier coefficients for f'. Performing integration by parts yields

   $$\pi a_n = \int_{-\pi}^{\pi} f(x)\cos nx dx = -\frac{1}{n}\int_{-\pi}^{\pi} f'(x)\sin nx dx .$$

   Therefore,

   $$\pi|a_n| \le \frac{1}{n}\int_{-\pi}^{\pi} |f'(x)| dx \to 0 , \quad n \to \infty .$$

   Similarly the same result holds for b_n.